1-1-2008

Cheap Boolean Role Constructors for Description Logics

Sebastian Rudolph

Markus Krotzsch

Pascal Hitzler
pascal.hitzler@wright.edu

Repository Citation
Rudolph, S., Krotzsch, M., & Hitzler, P. (2008). Cheap Boolean Role Constructors for Description Logics. *Lecture Notes in Computer Science, 5293*, 362-374.
https://corescholar.libraries.wright.edu/cse/138

Cheap Boolean Role Constructors for Description Logics
Technical Report

Sebastian Rudolph, Markus Krötzsch, and Pascal Hitzler

Institute AIFB, Universität Karlsruhe, Germany
{sru,mak,phi}@aifb.uni-karlsruhe.de

Abstract. We investigate the possibility of incorporating Boolean role constructors on simple roles into some of today's most popular description logics, focussing on cases where those extensions do not increase complexity of reasoning. We show that the expressive DLs $\mathcal{SHOIQ}$ and $\mathcal{SROIQ}$, serving as the logical underpinning of OWL and the forthcoming OWL 2, can accommodate arbitrary Boolean expressions. The prominent OWL-fragment $\mathcal{SHIQ}$ can be safely extended by safe role expressions, and the tractable fragments $\mathcal{EL}^{++}$ and DLP retain tractability if extended by conjunction on roles, where in the case of DLP the restriction on role simplicity can even be discarded.

1 Introduction

Research on description logics (DLs) is directed by two main goals: increasing expressivity while preserving desirable computational properties such as decidability (as a factual *conditio sine qua non*) and efficiency of reasoning, the latter qualitatively estimated in terms of worst-case complexities. These antagonistic dimensions gave rise to a great variety of logics: $\mathcal{SROIQ}$ and $\mathcal{SHOIQ}$ being of high expressiveness and complexity represent the one side of the spectrum, whereas the so called tractable fragments like $\mathcal{EL}^{++}$ and DLP provide lower expressivity yet allow for polynomial time reasoning.

In DL history, Boolean constructors (negation, conjunction, disjunction) on roles have occurred and have been investigated sporadically in many places, but have never been integrated into the mainstream of researched languages, let alone influenced standardisation considerations. In this paper we argue that Boolean constructors can – sometimes with appropriate restrictions – be incorporated into several of the most prominent DL languages, thereby significantly enhancing expressivity without increasing reasoning complexity.

To illustrate this gain in expressivity, we give some examples on the modelling capabilities of Boolean role constructors:

Universal role. A role U that connects all individuals of the described domain can e.g. be defined via $U \equiv \neg N$ as the complement of the empty role N, which in turn can be axiomatized by the GCI $\top \sqsubseteq \forall N.\bot$.

Role conjunction. This modelling feature comes in handy if certain non-tree-like properties (namely cases where two individuals are interconnected by more than one role) have to be described. The fact that somebody testifying against a relative is not put under oath can e.g. be formalised by $\exists(\mathit{testifiesAgainst} \sqcap \mathit{relativeOf}).\top \sqsubseteq \neg \mathit{UnderOath}$. Likewise, role conjunction allows for specifying disjointness of roles, as $\mathsf{Dis}(R, S)$ can be paraphrased as $\top \sqsubseteq \forall(R \sqcap S).\bot$.

Concept products. Thoroughly treated in [1], the concept product statement $C \times D \sqsubseteq R$ expresses that any instance of C is connected with any instance of D via role R. As an example, the fact that alkaline solutions neutralise acid solutions, which could expressed by the concept product axiom $\mathit{AlkalineSolution} \times \mathit{AcidSolution} \sqsubseteq \mathit{neutralises}$, can equivalently be stated by $\mathit{AlkalineSolution} \sqsubseteq \forall(\neg \mathit{neutralises}).\neg \mathit{AcidSolution}$ by using role negation.

Qualified role inclusion. Likewise, the specialisation of roles due to concept memberships of the involved individuals can be expressed. The rule-like FOL statement $C(x) \wedge R(x, y) \wedge D(y) \rightarrow S(x, y)$ (expressing that any C-instance and D-instance that are interconnected by R are also interconnected by S) can be cast into the GCI $C \sqsubseteq \forall(R \sqcap \neg S).\neg D$.

The rest of the paper is organised as follows. After providing the necessary definitions, we review existing work on Boolean role constructors. Then, we deal with the extension of $\mathcal{SROIQ}$ and $\mathcal{SHOIQ}$ by full Boolean role expressions on simple roles. Thereafter, we provide an according result for integrating safe Boolean role expressions into the description logic $\mathcal{SHIQ}$. The subsequent two sections settle the case for the tractable fragments $\mathcal{EL}^{++}$ and DLP, respectively, extending them by role conjunction. Finally, we conclude and elaborate on future work.

2 Preliminaries

In this section, we give the definition of the expressive description logic $\mathcal{SROIQB}_{\mathrm{s}}$ which is obtained from the well-known description logic $\mathcal{SROIQ}$ [2] by allowing arbitrary Boolean constructors on simple roles. We assume that the reader is familiar with description logics [3].

The DLs considered in this paper are based on four disjoint sets of *individual names* N_I, *concept names* N_C, and *simple role names* $\mathsf{N}_R^{\mathrm{s}}$ (containing the *universal role* $U \in \mathsf{N}_R$) as well as *non-simple role names* $\mathsf{N}_R^{\mathrm{n}}$. Furthermore, we let $\mathsf{N}_R \coloneqq \mathsf{N}_R^{\mathrm{s}} \cup \mathsf{N}_R^{\mathrm{n}}$.

Definition 1. *A $\mathcal{SROIQB}_{\mathrm{s}}$ Rbox for N_R is based on a set* $\mathbf{R}$ *of* atomic roles *defined as* $\mathbf{R} \coloneqq \mathsf{N}_R \cup \{R^- \mid R \in \mathsf{N}_R\}$, *where we set* $\mathrm{Inv}(R) \coloneqq R^-$ *and* $\mathrm{Inv}(R^-) \coloneqq R$ *to simplify notation. In turn, we distinguish* simple atomic roles $\mathbf{R}^{\mathrm{s}} \coloneqq \mathsf{N}_R^{\mathrm{s}} \cup \mathrm{Inv}(\mathsf{N}_R^{\mathrm{s}})$ *and* non-simple atomic roles $\mathbf{R}^{\mathrm{n}} \coloneqq \mathsf{N}_R^{\mathrm{n}} \cup \mathrm{Inv}(\mathsf{N}_R^{\mathrm{n}})$.

In the sequel, we will use the symbols R, S, possibly with subscripts, to denote atomic roles.

The set of Boolean role expressions $\mathbf{B}$ *is defined as follows:*

$$\mathbf{B} ::= \mathbf{R} \mid \neg\mathbf{B} \mid \mathbf{B} \sqcap \mathbf{B} \mid \mathbf{B} \sqcup \mathbf{B}.$$

The set $\mathbf{B}_\mathrm{s}$ *of* simple *role expressions comprises all those role expressions containing only simple role names. Moreover, a role expression will be called* safe, *if in its disjunctive normal form, every disjunct contains at least one non-negated role name.*

A generalised role inclusion axiom *(RIA) is a statement of the form* $V \sqsubseteq W$ *with simple role expressions* V *and* W, *or of the form*

$$S_1 \circ \ldots \circ S_n \sqsubseteq R$$

where each S_i *is a simple role expression or a non-simple atomic role, and where* R *is a non-simple atomic role. A set of such RIAs will be called a generalised* role hierarchy. *A role hierarchy is* regular *if there is a strict partial order* $\prec$ *on the non-simple roles* $\mathbf{R}^\mathrm{n}$ *such that*

- $S \prec R$ *iff* $\mathrm{Inv}(S) \prec R$, *and*
- *every RIA is of one of the forms*
  - $R \circ R \sqsubseteq R$,
  - $R^- \sqsubseteq R$,
  - $S_1 \circ \ldots \circ S_n \sqsubseteq R$,
  - $R \circ S_1 \circ \ldots \circ S_n \sqsubseteq R$,
  - $S_1 \circ \ldots \circ S_n \circ R \sqsubseteq R$,

  such that $R \in \mathsf{N}_R$ *is a (non-inverse) role name, and* $S_i \prec R$ *for* $i = 1, \ldots, n$ *whenever* S_i *is non-simple.*

A role assertion *is a statement of the form* $\mathsf{Ref}(R)$ *(*reflexivity*),* $\mathsf{Asy}(V)$ *(*asymmetry*), or* $\mathsf{Dis}(V, W)$ *(*role disjointness*), where* V *and* W *are simple role expressions, and* R *is a simple role expression or a non-simple role. A* $\mathcal{SROIQB}_\mathrm{s}$ Rbox *is the union of a set of role assertions together with a role hierarchy. A* $\mathcal{SROIQB}_\mathrm{s}$ *Rbox is regular if its role hierarchy is regular.*

Definition 2. *Given a* $\mathcal{SROIQB}_\mathrm{s}$ *Rbox* $\mathcal{R}$, *the set of* concept expressions $\mathbf{C}$ *is defined as follows:*

- $\mathsf{N}_C \subseteq \mathbf{C}$, $\top \in \mathbf{C}$, $\bot \in \mathbf{C}$,
- *if* $C, D \in \mathbf{C}$, $R \in \mathbf{R}$ *a simple role expression or non-simple role,* $V \in \mathbf{B}_\mathrm{s}$ *a simple role expression,* $a \in \mathsf{N}_I$, *and* n *a non-negative integer, then* $\neg C$, $C \sqcap D$, $C \sqcup D$, $\{a\}$, $\forall R.C$, $\exists R.C$, $\exists V.\mathsf{Self}$, ${\leq}n\,V.C$, *and* ${\geq}n\,V.C$ *are also concept expressions.*

Throughout this paper, the symbols C, D *will be used to denote concept expressions. A* $\mathcal{SROIQB}_\mathrm{s}$ Tbox *is a set of* general concept inclusion axioms *(GCIs) of the form* $C \sqsubseteq D$.

An individual assertion *can have any of the following forms:* $C(a)$, $R(a, b)$, $\neg S(a, b)$, $a \not\approx b$, *with* $a, b \in \mathsf{N}_I$ *individual names,* $C \in \mathbf{C}$ *a concept expression, and* $R, S \in \mathbf{R}$ *roles with* S *simple. A* $\mathcal{SROIQB}_\mathrm{s}$ Abox *is a set of individual assertions.*

A $\mathcal{SROIQB}_\mathrm{s}$ knowledge base KB *is the union of a regular Rbox* $\mathcal{R}$, *and an Abox* $\mathcal{A}$ *and Tbox* $\mathcal{T}$ *for* $\mathcal{R}$.

We further give the semantics of $\mathcal{SROIQB}_\mathrm{s}$ knowledge bases.

Name	Syntax	Semantics
inverse role	R^-	$\{(x,y) \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid (y,x) \in R^\mathcal{I}\}$
universal role	U	$\Delta^\mathcal{I} \times \Delta^\mathcal{I}$
role negation	$\neg V$	$\{(x,y) \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid (x,y) \notin R^\mathcal{I}\}$
role conjunction	$V \sqcap W$	$V^\mathcal{I} \cap W^\mathcal{I}$
role disjunction	$V \sqcup W$	$V^\mathcal{I} \cup W^\mathcal{I}$
top	$\top$	$\Delta^\mathcal{I}$
bottom	$\bot$	$\emptyset$
negation	$\neg C$	$\Delta^\mathcal{I} \setminus C^\mathcal{I}$
conjunction	$C \sqcap D$	$C^\mathcal{I} \cap D^\mathcal{I}$
disjunction	$C \sqcup D$	$C^\mathcal{I} \cup D^\mathcal{I}$
nominals	$\{a\}$	$\{a^\mathcal{I}\}$
univ. restriction	$\forall R.C$	$\{x \in \Delta^\mathcal{I} \mid (x,y) \in R^\mathcal{I}$ implies $y \in C^\mathcal{I}\}$
exist. restriction	$\exists R.C$	$\{x \in \Delta^\mathcal{I} \mid$ for some $y \in \Delta^\mathcal{I}$, $(x,y) \in R^\mathcal{I}$ and $y \in C^\mathcal{I}\}$
Self concept	$\exists S.\mathsf{Self}$	$\{x \in \Delta^\mathcal{I} \mid (x,x) \in S^\mathcal{I}\}$
qualified number	$\leq n\, S.C$	$\{x \in \Delta^\mathcal{I} \mid \#\{y \in \Delta^\mathcal{I} \mid (x,y) \in S^\mathcal{I}$ and $y \in C^\mathcal{I}\} \leq n\}$
restriction	$\geq n\, S.C$	$\{x \in \Delta^\mathcal{I} \mid \#\{y \in \Delta^\mathcal{I} \mid (x,y) \in S^\mathcal{I}$ and $y \in C^\mathcal{I}\} \geq n\}$

Fig. 1. Semantics of concept constructors in $\mathcal{SROIQB}_\mathrm{s}$ for an interpretation $\mathcal{I}$ with domain $\Delta^\mathcal{I}$.

Definition 3. *An interpretation $\mathcal{I}$ consists of a set $\Delta^\mathcal{I}$ called* domain *(the elements of it being called* individuals*) together with a function $\cdot^\mathcal{I}$ mapping*

- *individual names to elements of $\Delta^\mathcal{I}$,*
- *concept names to subsets of $\Delta^\mathcal{I}$, and*
- *role expressions to subsets of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$.*

The function $\cdot^\mathcal{I}$ is inductively extended to role and concept expressions as shown in Table 1. An interpretation $\mathcal{I}$ satisfies *an axiom φ if we find that $\mathcal{I} \models \varphi$:*

- *$\mathcal{I} \models V \sqsubseteq W$ if $V^\mathcal{I} \subseteq W^\mathcal{I}$,*
- *$\mathcal{I} \models V_1 \circ \ldots \circ V_n \sqsubseteq R$ if $V_1^\mathcal{I} \circ \ldots \circ V_n^\mathcal{I} \sqsubseteq R^\mathcal{I}$ ($\circ$ being overloaded to denote the standard composition of binary relations here),*
- *$\mathcal{I} \models \mathsf{Ref}(R)$ if $R^\mathcal{I}$ is a reflexive relation,*
- *$\mathcal{I} \models \mathsf{Asy}(V)$ if $V^\mathcal{I}$ is antisymmetric and irreflexive,*
- *$\mathcal{I} \models \mathsf{Dis}(V,W)$ if $V^\mathcal{I}$ and $W^\mathcal{I}$ are disjoint,*
- *$\mathcal{I} \models C \sqsubseteq D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$.*

An interpretation $\mathcal{I}$ satisfies *a knowledge base* KB *(we then also say that $\mathcal{I}$ is a* model *of* KB *and write $\mathcal{I} \models$* KB*) if it satisfies all axioms of* KB*. A knowledge base* KB *is* satisfiable *if it has a model. Two knowledge bases are* equivalent *if they have exactly the same models, and they are* equisatisfiable *if either both are unsatisfiable or both are satisfiable.*

We obtain $\mathcal{SROIQ}$ from $\mathcal{SROIQB}_\mathrm{s}$ by disallowing all junctors in role expressions. Further details on $\mathcal{SROIQ}$ can be found in [2]. We have omitted here several syntactic constructs that can be expressed indirectly, especially role assertions for transitivity, reflexivity of simple roles, and symmetry. Moreover, the DL $\mathcal{SHOIQ}$ is obtained from

$\mathcal{SROIQ}$ by discarding the universal role as well as reflexivity, asymmetry, role disjointness statements and allowing only RIAs of the form $R \sqsubseteq S$ or $R \circ R \sqsubseteq R$. Further DLs will be defined as they occur.

3 Related Work

Boolean constructors on roles have been investigated in the context of both description and modal logics. [4] for used them extensively for the definition of a DL that is equivalent to the two-variable fragment of FOL.

As a classical result on complexities, it was shown in [5], that augmenting $\mathcal{ALC}$ with full Boolean role constructors ($\mathcal{ALCB}$) leads to NExpTime-completeness of the standard reasoning tasks (while restricting to role negation [5] or role conjunction [6] only retains ExpTime-completeness). This complexity does not further increase when allowing for inverses, qualified number restrictions, and nominals as was shown in [6] by a polynomial translation of $\mathcal{ALCIQB}$ into C^2, the two variable fragment of first order logic with counting quantifiers, which in turn was proven to be NExpTime-complete in [7]. Also the recently considered description logic $\mathcal{ALBO}$ [8] falls in that range of NExpTime-complete DLs.

On the contrary, it was also shown in [6] that restricting to *safe* Boolean role constructors keeps $\mathcal{ALC}$'s reasoning complexity in ExpTime, even when adding inverses and qualified number restrictions ($\mathcal{ALCI}b$).

For logics including modelling constructs that deal with role concatenation like transitivity or – more general – complex rule inclusion axioms, results on complexities in the presence of Boolean role constructors are more sparse. [9] shows that $\mathcal{ALC}$ can be extended by negation and regular expressions on roles while keeping reasoning within ExpTime. Furthermore, [10] provided ExpTime complexity for a similar logic that includes inverses and qualified number restriction but reverts to safe negation on roles.

An extension of $\mathcal{SHIQ}$ with role conjunction (denoted $\mathcal{SHIQ}^{\sqcap}$) is presented in [11] in the context of conjunctive query answering, the results implying an upper bound of 2ExpTime.

4 $\mathcal{SROIQB}_s$ and $\mathcal{SHOIQB}_s$

In this section, we show that adding arbitrary (i.e. also unsafe) Boolean role expressions to the widely known description logics $\mathcal{SROIQ}$ and $\mathcal{SHOIQ}$ does not harm their reasoning complexities – N2ExpTime [12] and NExpTime [6], repectively – if this extension is restricted to simple roles.

Note that in the sequel, $\mathcal{SHOIQ}$ (resp. $\mathcal{SHOIQB}_s$) will be treated as a special case of $\mathcal{SROIQ}$ (resp. $\mathcal{SROIQB}_s$), as most considerations hold for both cases.

As shown in [12], any $\mathcal{SROIQ}$ ($\mathcal{SHOIQ}$) knowledge base can be transformed into an equisatisfiable knowledge base containing only axioms of the form:

Table 1. Additional transformation for $\mathcal{SROIQB}_s$ and $\mathcal{SHOIQB}_s$. A, B are concept names. V, W are simple role expressions. V_i are simple role expressions or non-simple roles. $\hat{V}$ is a simple role expression that is not just a role. R is a non-simple role name. $\overline{S}$ is a new simple role name.

$A \sqsubseteq \forall\hat{V}.B$	$\mapsto$	$\{A \sqsubseteq \forall\overline{S}.B, \hat{V} \sqsubseteq \overline{S}\}$
$A \sqsubseteq {\geq}n\,\hat{V}.B$	$\mapsto$	$\{A \sqsubseteq {\geq}n\,\overline{S}.B, \overline{S} \sqsubseteq \hat{V}\}$
$A \sqsubseteq {\leq}n\,\hat{V}.B$	$\mapsto$	$\{A \sqsubseteq {\leq}n\,\overline{S}.B, \hat{V} \sqsubseteq \overline{S}\}$
$A \sqsubseteq \exists\hat{V}.\mathsf{Self}$	$\mapsto$	$\{A \sqsubseteq \exists\overline{S}.\mathsf{Self}, \overline{S} \sqsubseteq \hat{V}\}$
$\mathsf{Dis}(V, W)$	$\mapsto$	$\{V \sqcap W \sqsubseteq \overline{S}, \top \sqsubseteq \forall\overline{S}.\bot\}$
$V_1 \circ \ldots \circ \hat{V} \circ \ldots \circ V_n \sqsubseteq R$	$\mapsto$	$\{V_1 \circ \ldots \circ \overline{S} \circ \ldots \circ V_n \sqsubseteq R, \hat{V} \sqsubseteq \overline{S}\}$

$$
\begin{array}{lll}
A \sqsubseteq \forall R.B & \bigsqcap A_i \sqsubseteq \bigsqcup B_j & S_1 \sqsubseteq S_2 \\
A \sqsubseteq {\geq}n\,S.B & A \equiv \{a\} & S_1 \sqsubseteq S_2^- \\
A \sqsubseteq {\leq}n\,S.B & A \equiv \exists S.\mathsf{Self} & \mathsf{Dis}(S_1, S_2)
\end{array}
$$
$$R_1 \circ \ldots \circ R_n \sqsubseteq R.$$

Trivially, this normalization can be applied to $\mathcal{SROIQB}_s$ ($\mathcal{SHOIQB}_s$) as well, yielding the same types of axioms whereas simple role expressions may occur in the place of simple roles. A second transformation carried out by exhaustively applying the transformation steps depicted in Table 1 yields an equisatisfiable knowledge base containing only the original axiom types depicted above (i.e. again only simple role names in places of $S_{(i)}$ and role names in places of R_i) and just one additional axiom type $W \sqsubseteq V$ with W,V simple role expressions. As shown in [12], any of these original axiom types except the one containing role concatenation can be translated into C^2, the two-variable fragment of first order logic with counting quantifiers. The additionally introduced type of axiom can clearly also be transformed into C^2 statements namely into the proposition $\forall xy(\Phi(W) \to \Phi(V))$ where Φ is inductively defined by:

$$
\begin{aligned}
\Phi(S) &= S(x, y) \\
\Phi(S^-) &= S(y, x) \\
\Phi(\neg V) &= \neg\Phi(V) \\
\Phi(V \sqcap W) &= \Phi(V) \wedge \Phi(W) \\
\Phi(V \sqcup W) &= \Phi(V) \vee \Phi(W)
\end{aligned}
$$

Further following the argumentation from [12], the remaining complex role inclusions not directly convertible into C^2 can be taken into account by cautiously materializing the consequences resulting from their interplay with axioms of the type $A \sqsubseteq \forall R.B$ through automata encoding techniques – see also [13]. This way, one obtains a C^2 theory that is satisfiable exactly if the original knowledge base is. In the case of $\mathcal{SROIQ}$ (and hence $\mathcal{SROIQB}_s$), this can result in an exponential blowup of the knowledge base while for $\mathcal{SHOIQB}_s$ (and hence $\mathcal{SHOIQ}$) the transformation is time polynomial. Thus we see that the upper complexity bounds for $\mathcal{SROIQ}$ and $\mathcal{SHOIQ}$ carry over to $\mathcal{SROIQB}_s$ and $\mathcal{SHOIQB}_s$ by just a slight extension of the according proofs from [12] while the lower bounds follow directly from those of $\mathcal{SROIQ}$ and $\mathcal{SHOIQ}$. Hence, we can establish the following theorem.

Theorem 1. *Satisfiability checking, instance retrieval, and computing class subsumptions for $\mathcal{SROIQB}_s$ ($\mathcal{SHOIQB}_s$) knowledge bases is* N2ExpTime*-complete (*NExpTime*-complete).*

While the results established in this section are rather straightforward consequences of known results, their implications for practice might be more significant: they show that the DLs underlying OWL and OWL 2 can be extended by arbitrary Boolean constructors on simple roles without increasing the worst case complexity of reasoning.

5 $\mathcal{SHIQ}b_s$

$\mathcal{SHIQ}$ is a rather expressive fragment obtained from $\mathcal{SHOIQ}$ by disallowing nominals, where (in contrast to the NExpTime-complete $\mathcal{SHOIQ}$) reasoning is known to be ExpTime-complete [6].

In this section we will introduce the extension of $\mathcal{SHIQ}$ by safe role expressions on simple roles. Thereafter, we will present a technique for removing transitivity statements from $\mathcal{SHIQ}b_s$ knowledge bases in a satisfiability preserving way. This yields two results: on the one hand, we provide a way how existing reasoning procedures for $\mathcal{ALCI}b$ like e.g. those described in [6, 10, 14] can be used to solve $\mathcal{SHIQ}b_s$ reasoning tasks. On the other hand, as the transformation procedure can be done in polynomial time, the known upper bound for the complexity of reasoning in $\mathcal{ALCI}b$ – namely ExpTime – carries over to $\mathcal{SHIQ}b_s$.

Definition 4. *A $\mathcal{SHIQ}b_s$ knowledge base is a $\mathcal{SHOIQB}_s$ knowledge base that contains no nominals and only safe role expressions.*

Based on a fixed knowledge base KB*, we define $\sqsubseteq^*$ as the smallest binary relation on the non-simple atomic roles $\mathbf{R}_n$ such that:*

- $R \sqsubseteq^* R$ *for every atomic role R,*
- $R \sqsubseteq^* S$ *and* $\mathrm{Inv}(R) \sqsubseteq^* \mathrm{Inv}(S)$ *for every Rbox axiom $R \sqsubseteq S$, and*
- $R \sqsubseteq^* T$ *whenever* $R \sqsubseteq^* S$ *and* $S \sqsubseteq^* T$.

Given an atomic non-simple role R, we write $\mathsf{Trans}(R) \in \mathrm{KB}$ *as an abbreviation for:* $R \circ R \sqsubseteq R \in \mathrm{KB}$ *or* $\mathrm{Inv}(R) \circ \mathrm{Inv}(R) \sqsubseteq \mathrm{Inv}(R) \in \mathrm{KB}$.

Slightly generalising according results from [15], we now show that any $\mathcal{SHIQ}b_s$ knowledge base can be transformed into an equisatisfiable knowledge base not containing transitivity statements.

Definition 5. *Given a $\mathcal{SHIQ}b_s$ knowledge base* KB*, let* $\mathsf{clos}(\mathrm{KB})$ *denote the smallest set of concept expressions where*

- $\mathsf{NNF}(\neg C \sqcup D) \in \mathsf{clos}(\mathrm{KB})$ *for any Tbox axiom $C \sqsubseteq D$,*
- $D \in \mathsf{clos}(\mathrm{KB})$ *for every subexpression D of some concept* $C \in \mathsf{clos}(\mathrm{KB})$,
- $\mathsf{NNF}(\neg C) \in \mathsf{clos}(\mathrm{KB})$ *for any* $\leq n\, R.C \in \mathsf{clos}(\mathrm{KB})$,

– $\forall S.C \in \mathsf{clos}(\mathrm{KB})$ *whenever* $\mathsf{Trans}(S) \in \mathrm{KB}$ *and* $S \sqsubseteq^* R$ *for a role* R *with* $\forall R.C \in \mathsf{clos}(\mathrm{KB})$.

Moreover, let $\Omega(\mathrm{KB})$ *denote the knowledge base obtained from* KB *by*

– *removing all transitivity axioms* $R \circ R \sqsubseteq R$ *and*
– *adding the axiom* $\forall R.C \sqsubseteq \forall S.(\forall S.C)$ *for every* $\forall R.C \in \mathsf{clos}(\mathrm{KB})$ *with* $\mathsf{Trans}(S) \in \mathrm{KB}$ *and* $S \sqsubseteq^* R$.

Proposition 1. *Let* KB *ba a* $\mathcal{SHIQ}b_{\mathrm{s}}$ *knowledge base. Then,* KB *and* $\Omega(\mathrm{KB})$ *are equisatisfiable.*

Proof. Obviously, we have that $\mathrm{KB} \models \Omega(\mathrm{KB})$, hence every model of KB is a model of $\Omega(\mathrm{KB})$ as well.

For the other direction, let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a model of $\Omega(\mathrm{KB})$. Then we define a new interpretation $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ as follows:

– $\Delta^{\mathcal{J}} := \Delta^{\mathcal{I}}$
– $a^{\mathcal{J}} := a^{\mathcal{I}}$ for every $a \in \mathsf{N}_I$
– $A^{\mathcal{J}} := A^{\mathcal{I}}$ for every $A \in \mathsf{N}_C$
– for all simple roles R, we set $R^{\mathcal{J}} := R^{\mathcal{I}}$
– for all non-simple roles R, $R^{\mathcal{J}}$ is set to the transitive closure of $R^{\mathcal{I}}$ if $\mathsf{Trans}(R) \in \mathrm{KB}$, otherwise $R^{\mathcal{J}} := R^{\mathcal{I}} \cup \bigcup_{S \sqsubseteq^* R, S \in \mathsf{N}_R^{\mathrm{n}}} S^{\mathcal{J}}$

As a direct consequence of this definition, note that for all simple concept expressions $V \in \mathbf{B}_{\mathrm{s}}$ we have $V^{\mathcal{J}} = V^{\mathcal{I}}$ (fact †).

We now prove that $\mathcal{J}$ is a model of KB by considering all axioms starting with the Rbox: Firstly, every transitivity axiom of KB is obviously fulfilled by definition of $\mathcal{J}$. Secondly, every role inclusion $V \sqsubseteq W$ axiom is fulfilled:

If both V and W are simple role expressions this is a trivial consequence from (†). If V is a simple role expression and W is a non-simple role, this follows from (†) and the fact that by construction of $\mathcal{J}$, for every non-simple role R holds $R^{\mathcal{I}} \subseteq R^{\mathcal{J}}$.

It remains the case that both V and W are non-simple roles. In case W is not transitive, this follows directly from the definition, otherwise we can conclude it from the fact that the transitive closure is a monotone operation w.r.t. set inclusion.

We proceed by examining the concept expressions $C \in \mathsf{clos}(\mathrm{KB})$ and show via structural induction that $C^{\mathcal{I}} \subseteq C^{\mathcal{J}}$. As base case, for every concept of the form A, or $\neg A$ for $A \in \mathsf{N}_C$ this claim follows directly from the definition of $\mathcal{J}$. We proceed with the induction steps for all possible forms of a complex concept C (mark that all $C \in \mathsf{clos}(\mathrm{KB})$ are in negation normal form):

– Clearly, if $D_1^{\mathcal{I}} \subseteq D_1^{\mathcal{J}}$ and $D_2^{\mathcal{I}} \subseteq D_2^{\mathcal{J}}$ by induction hypothesis, we can directly conclude $(D_1 \sqcap D_2)^{\mathcal{I}} \subseteq (D_1 \sqcap D_2)^{\mathcal{J}}$ as well as $(D_1 \sqcup D_2)^{\mathcal{I}} \subseteq (D_1 \sqcup D_2)^{\mathcal{J}}$.
– Likewise, as we have $V^{\mathcal{I}} \subseteq V^{\mathcal{J}}$ for all simple role expressions and non-simple roles V and again $D^{\mathcal{I}} \subseteq D^{\mathcal{J}}$ due to the induction hypothesis, we can conclude $(\exists V.D)^{\mathcal{I}} \subseteq (\exists V.D)^{\mathcal{J}}$ as well as $(\geq n\, V.D)^{\mathcal{I}} \subseteq (\geq n\, V.D)^{\mathcal{J}}$.

– Now, consider a $C = \forall V.D$. If V is a simple role expression, we know that $V^{\mathcal{J}} = V^{\mathcal{I}}$, whence we can derive $(\forall V.D)^{\mathcal{I}} \subseteq (\forall V.D)^{\mathcal{J}}$ from the induction hypothesis.

  It remains to consider the case $C = \forall R.D$ for non-simple roles R. Assume $\delta \in (\forall R.D)^{\mathcal{I}}$. If there is no δ' with $(\delta, \delta') \in R^{\mathcal{J}}$, then $\delta \in (\forall R.D)^{\mathcal{J}}$ is trivially true. Now assume there are such δ'. For each of them, we can distinguish two cases:

  - $(\delta, \delta') \in R^{\mathcal{J}}$, implying $\delta' \in D^{\mathcal{I}}$ and, via the induction hypothesis, $\delta' \in D^{\mathcal{J}}$,
  - $(\delta, \delta') \notin R^{\mathcal{J}}$. Yet, by construction of $\mathcal{J}$, this means that there is a role S with $S \sqsubseteq^* R$ and $\mathsf{Trans}(S) \in \mathrm{KB}$ and a sequence $\delta = \delta_0, \ldots, \delta_n = \delta'$ with $(\delta_k, \delta_{k+1}) \in S^{\mathcal{I}}$ for all $0 \leq k < n$. By definition of Ω, the knowledge base $\Omega(\mathrm{KB})$ contains the axiom $\forall R.D \sqsubseteq \forall S.(\forall S.D)$, hence we have $\delta \in \forall S.(\forall S.D)$ wherefrom a simple inductive argument ensures $\delta_k \in D^{\mathcal{I}}$ for all δ_k including $\delta_n = \delta'$.

  So we can conclude that for all such δ' we have $\delta' \in D^{\mathcal{I}}$. Via the induction hypothesis follows $\delta \in D^{\mathcal{J}}$ and hence we can conclude $\delta \in (\forall R.D)^{\mathcal{J}}$.

– Finally, consider $C = \leq n\, R.D$ and assume $\delta \in (\leq n\, R.D)^{\mathcal{I}}$. From the fact that R must be simple follows $R^{\mathcal{J}} = R^{\mathcal{I}}$. Moreover, since both D and $\mathsf{NNF}(\neg D)$ are contained in $\mathsf{clos}(\mathrm{KB})$ the induction hypothesis gives $D^{\mathcal{J}} = D^{\mathcal{I}}$. Those two facts together directly imply $\delta \in (\leq n\, R.D)^{\mathcal{I}}$.

Now considering an arbitrary KB Tbox axiom $C \sqsubseteq D$, we find $(\mathsf{NNF}(\neg C) \sqcup D)^{\mathcal{I}} = \Delta^{\mathcal{I}}$ as $\mathcal{I}$ is a model of KB. Moreover – by the correspondence just shown – we have $(\mathsf{NNF}(\neg C) \sqcup D)^{\mathcal{I}} \subseteq (\mathsf{NNF}(\neg C) \sqcup D)^{\mathcal{I}}$ and hence also $(\mathsf{NNF}(\neg C) \sqcup D)^{\mathcal{J}} = \Delta^{\mathcal{J}}$ making $C \sqsubseteq D$ an axiom satisfied in $\mathcal{J}$. This finishes the proof. □

Taking into account that the presented transformation is time polynomial, this result can now be employed to determine the complexity of $\mathcal{SHIQ}b_s$.

Theorem 2. *Reasoning in $\mathcal{SHIQ}b_s$ is* ExpTime*-complete.*

Proof. Clearly, all standard reasoning problems can be reduced to knowledge base satisfiability checking as usual.

Now, by Proposition 1, any given $\mathcal{SHIQ}b_s$ knowledge base KB can be transformed into an $\mathcal{ALCHI}b$ knowledge base $\Omega(\mathrm{KB})$ in polynomial time. Furthermore, all role inclusion axioms can be removed from $\Omega(\mathrm{KB})$ as follows. First, all role names contained in $\Omega(\mathrm{KB})$ can be declared to be simple without violating the syntactic constraints. Second, every role inclusion axiom $V \sqsubseteq W$ (with V, W being safe by definition) can be equivalently transformed into the GCI $\top \sqsubseteq \forall(V \sqcap \neg W).\bot$. Note that then $V \sqcap \neg W$ is safe as well and therefore admissible. Moreover the transformation is obviously time linear. So we end up with an $\mathcal{ALCI}b$ knowledge base whose satisfiability checking is ExpTime-complete due to [6]. □

So we have shown that allowing safe Boolean expressions on simple roles does not increase the ExpTime reasoning complexity of $\mathcal{SHIQ}$. On the other hand, the recent results on $\mathcal{SHIQ}^{\sqcap}$ [11] seem to indicate that the role simplicity condition is essential for staying within ExpTime even though no definite hardness result for general $\mathcal{SHIQ}^{\sqcap}$ was provided. The safety condition on role expressions, in turn, is clearly needed: dropping it would lead to a DL comprising $\mathcal{ALCB}$which is known to be NExpTime-complete [5].

6 $\mathcal{EL}^{++}(\sqcap_s)$

In this section, we investigate role conjunction for the DL $\mathcal{EL}^{++}$ [16], for which many typical inference problems can be solved in polynomial time. We simplify our presentation by omitting concrete domains from $\mathcal{EL}^{++}$ – they are not affected by our extension and can be treated as shown in [16].

Definition 6. *An atomic role of $\mathcal{EL}^{++}(\sqcap_s)$ is a (non-inverse) role name. An $\mathcal{EL}^{++}(\sqcap_s)$ role expression is a simple role expression containing only role conjunction. An $\mathcal{EL}^{++}(\sqcap_s)$ Rbox is a set of generalised role inclusion axioms (using $\mathcal{EL}^{++}(\sqcap_s)$role expressions and non-simple atomic roles), and an $\mathcal{EL}^{++}(\sqcap_s)$ Tbox is a $\mathcal{SROIQB}_s$ Tbox that contains only the concept constructors: $\sqcap$, $\exists$, $\top$, $\bot$ and only $\mathcal{EL}^{++}(\sqcap_s)$role expressions.*

Note that we do not have any requirement for regularity of roles but we have to introduce the notion of role simplicity in the context of $\mathcal{EL}^{++}$. In a first step, we observe that any $\mathcal{EL}^{++}(\sqcap_s)$ knowledge base can be converted into a normal form.

Definition 7. *An $\mathcal{EL}^{++}(\sqcap_s)$ knowledge base* KB *is in* normal form *if it contains only axioms of one of the following forms:*

$$\begin{array}{rclcrclcrcl} A & \sqsubseteq & C & \quad & A \sqcap B & \sqsubseteq & C & \quad & R & \sqsubseteq & T \\ \exists R.A & \sqsubseteq & B & & A & \sqsubseteq & \exists R.B & & R \circ S & \sqsubseteq & T \\ & & & & R \sqcap S & \sqsubseteq & T & & & & \end{array}$$

where $A, B \in \mathsf{N}_C \cup \{\{a\} \mid a \in \mathsf{N}_I\} \cup \{\top\}$, $C \in \mathsf{N}_C \cup \{\{a\} \mid a \in \mathsf{N}_I\} \cup \{\bot\}$, and $R, S, T \in \mathsf{N}_R$.

Proposition 2. *Any $\mathcal{EL}^{++}(\sqcap_s)$ knowledge base can be transformed into an equisatisfiable $\mathcal{EL}^{++}(\sqcap_s)$ knowledge base in normal form. The transformation can be done in linear time.*

Proof. The transformation is accomplished by the rules of Table 2, where each rule describes the replacement of some axiom by one or more alternative axioms. In every of the five steps, the corresponding rules are applied exhaustively to the knowledge base. Polynomiality of this conversion can then be shown in analogy to the normal form transformation given in [16]. □

Subsequently, we show that the only axiom type of this normal form not covered by $\mathcal{EL}^{++}$ can be removed from an $\mathcal{EL}^{++}(\sqcap_s)$ knowledge base while preserving satisfiability if the relevant consequences are materialized before.

Definition 8. *Given an $\mathcal{EL}^{++}(\sqcap_s)$ knowledge base* KB *in normal form, let $\Theta^{\sqcap}$*(KB) *denote the knowledge base obtained from* KB *by*

- *adding $R_1 \sqsubseteq R_2$ for all $R_2 \in R_1^{\sqsubseteq}$ where $S^{\sqsubseteq} \subseteq \mathsf{N}_R$ denotes the smallest set of role names containing S and satisfying*
  - *$T \in S^{\sqsubseteq}$, whenever $R \in S^{\sqsubseteq}$ and $R \sqsubseteq T \in$ KB as well as*
  - *$T \in S^{\sqsubseteq}$, whenever $R_1, R_2 \in S^{\sqsubseteq}$ and $R_1 \sqcap R_2 \sqsubseteq T \in$ KB,*
- *removing every axiom of the form $S_1 \sqcap S_2 \sqsubseteq R$ and instead adding the axioms $\exists S_1.\{o\} \sqcap \exists S_2.\{o\} \sqsubseteq \exists R.\{o\}$ for every individual name $\{o\}$.*

Table 2. Normal form transformation for $\mathcal{EL}^{++}(\sqcap_s)$. $A, B, C, \hat{A}, \hat{C}$, and D are concept expressions, where $\hat{A}$ and $\hat{C}$ are neither concept names nor nominals. V_i, W are simple concept expressions or non-simple role names, while $\hat{V}$, $\hat{W}$ are simple concept expressions that are not role names. R, R_i are simple role name, S, and T are non-simple role names. Every overlined role or concept name is fresh. Commutativity and associativity of $\sqcap$ (for both concepts and roles) is assumed to simplify the rule set.

P1:	$V_1 \circ \ldots \circ V_{n-1} \circ V_n \sqsubseteq R$	$\mapsto$	$\{V_1 \circ \ldots \circ V_{n-1} \sqsubseteq \overline{T}, \overline{T} \circ V_n \sqsubseteq R\}$
	$B \sqcap \hat{A} \sqsubseteq C$	$\mapsto$	$\{\hat{A} \sqsubseteq \overline{D}, \overline{D} \sqcap B \sqsubseteq C\}$
	$\exists V.\hat{A} \sqsubseteq B$	$\mapsto$	$\{\hat{A} \sqsubseteq \overline{D}, \exists V.\overline{D} \sqsubseteq B\}$
	$\bot \sqsubseteq C$	$\mapsto$	$\emptyset$
P2:	$A \sqsubseteq B \sqcap C$	$\mapsto$	$\{A \sqsubseteq B, A \sqsubseteq C\}$
	$\hat{A} \sqsubseteq \hat{C}$	$\mapsto$	$\{\hat{A} \sqsubseteq \overline{D}, \overline{D} \sqsubseteq \hat{C}\}$
	$A \sqsubseteq \exists V.\hat{C}$	$\mapsto$	$\{A \sqsubseteq \exists V.\overline{D}, \overline{D} \sqsubseteq \hat{C}\}$
	$A \sqsubseteq \top$	$\mapsto$	$\emptyset$
P3:	$A \sqsubseteq \exists(R_1 \sqcap \ldots \sqcap R_n).B$	$\mapsto$	$\{A \sqsubseteq \exists\overline{R}.B, \overline{R} \sqsubseteq R_i \mid 1 \leq i \leq n\}$
	$\exists\hat{V}.A \sqsubseteq B$	$\mapsto$	$\{\exists\overline{R}.A \sqsubseteq B, \hat{V} \sqsubseteq \overline{R}\}$
	$\hat{V} \circ W \sqsubseteq S$	$\mapsto$	$\{\hat{V} \sqsubseteq \overline{R}, \overline{R} \circ W \sqsubseteq S\}$
	$W \circ \hat{V} \sqsubseteq S$	$\mapsto$	$\{\hat{V} \sqsubseteq \overline{R}, W \circ \overline{R} \sqsubseteq S\}$
P4:	$\hat{V} \sqsubseteq \hat{W}$	$\mapsto$	$\{\hat{V} \sqsubseteq \overline{R}, \overline{R} \sqsubseteq \hat{W}\}$
	$R \sqsubseteq S_1 \sqcap \ldots \sqcap S_m$	$\mapsto$	$\{R \sqsubseteq S_i \mid 1 \leq i \leq m\}$
P5:	$R_1 \sqcap \ldots \sqcap R_{n-1} \sqcap R_n \sqsubseteq S$	$\mapsto$	$\{R_1 \sqcap \ldots \sqcap R_{n-1} \sqsubseteq \overline{R}, \overline{R} \sqcap R_n \sqsubseteq S\}$

Note that $\Theta^{\sqcap}(\mathrm{KB})$ can be computed in polynomial time. In particular, finding the closed sets $R^{\sqsubseteq}$ can be done in linear time w.r.t. the size of KB, e.g. using the *linclosure* algorithm from [17].

Proposition 3. *Let* KB *be an* $\mathcal{EL}^{++}(\sqcap_s)$ *knowledge base. Then,* KB *and* $\Theta^{\sqcap}(\mathrm{KB})$ *are equisatisfiable.*

Proof. First, note that any model of KB is a model of $\Theta^{\sqcap}(\mathrm{KB})$ as all the axioms in $\Theta^{\sqcap}(\mathrm{KB}) \setminus \mathrm{KB}$ are consequences from KB.

Second, assume $\Theta^{\sqcap}(\mathrm{KB})$ is satisfiable. We now use an arbitrary model $\mathcal{I}$ of $\Theta^{\sqcap}(\mathrm{KB})$ to construct an interpretation $\mathcal{J}$ as follows (where we let N'_R denote the simple role names occurring in KB and $\mathsf{N}_I{}^{\mathcal{I}} := \{o^{\mathcal{I}} \mid o \in \mathsf{N}_I\}$):

- $\Delta^{\mathcal{J}} := \mathsf{N}_I{}^{\mathcal{I}} \cup ((\mathsf{N}^s_R \cup \{\emptyset\}) \times (\Delta^{\mathcal{I}} \setminus \mathsf{N}_I{}^{\mathcal{I}}))$. I.e., every unnamed individual δ from $\Delta^{\mathcal{I}}$ is substituted by copies endowed with the simple role names and one additional copy $(\emptyset, \delta)$. We will use the function $\mathsf{orig} : \Delta^{\mathcal{J}} \to \Delta^{\mathcal{I}}$ to refer to the "$\mathcal{I}$-origin" of an $\mathcal{J}$-individual by letting $\mathsf{orig}(o^{\mathcal{J}}) := o^{\mathcal{I}}$ for all $o \in \mathsf{N}_I$ as well as $\mathsf{orig}((x, \delta)) := \delta$ for every $(x, \delta) \in \Delta^{\mathcal{J}} \setminus \mathsf{N}_I{}^{\mathcal{I}}$.
- for $o \in \mathsf{N}_I$, let $o^{\mathcal{J}} := o^{\mathcal{I}}$
- for $\delta \in \Delta^{\mathcal{J}}$, let $\delta \in A^{\mathcal{J}}$ iff $\mathsf{orig}(\delta) \in A^{\mathcal{I}}$
- $(\delta, \epsilon) \in S^{\mathcal{J}}$, iff $(\mathsf{orig}(\delta), \mathsf{orig}(\epsilon)) \in S^{\mathcal{I}}$ and one of the following is the case:
  - $\epsilon = o^{\mathcal{I}}$ for some $o \in \mathsf{N}_I$, (*)

- S is non-simple, (**)
- $\epsilon = (R, \delta)$ for some $\delta \in \Delta^{\mathcal{I}} \setminus \mathsf{N}_I{}^{\mathcal{I}}$ and $S \in R^{\sqsubseteq}$. (***)

We now proceed by showing that $\mathcal{J}$ is a model of KB.

Clearly, $\mathcal{J}$ satisfies all axioms of the shape $A \sqsubseteq C$ and $A \sqcap B \sqsubseteq C$ since $\mathcal{I}$ does.

Considering the axiom type $\exists R.A \sqsubseteq B$, we observe the following for every $\delta \in (\exists R.A)^{\mathcal{J}}$: taking the witness $\epsilon \in \Delta^{\mathcal{J}}$ with $(\delta, \epsilon) \in R^{\mathcal{J}}$ and $\epsilon \in A^{\mathcal{J}}$, we find that $(\mathsf{orig}(\delta), \mathsf{orig}(\epsilon)) \in R^{\mathcal{I}}$ as well as $\mathsf{orig}(\epsilon) \in A^{\mathcal{I}}$, hence, $\mathsf{orig}(\delta) \in (\exists R.A)^{\mathcal{I}}$ and via the considered axiom $\mathsf{orig}(\delta) \in B^{\mathcal{I}}$, such that we can by construction conclude $\delta \in B^{\mathcal{J}}$.

For axioms of the type $A \sqsubseteq \exists R.B$, assume $\delta \in A^{\mathcal{J}}$ directly implying $\mathsf{orig}(\delta) \in A^{\mathcal{I}}$. By the considered axiom being in $\Theta^{\sqcap}(\mathrm{KB})$, we also find some $\eta \in \Delta^{\mathcal{I}}$ for which $(\mathsf{orig}(\delta), \eta) \in R^{\mathcal{I}}$ and $\eta \in A^{\mathcal{I}}$. First, observe that any $\eta' \in \Delta^{\mathcal{J}}$ originating from η satisfies $\eta' \in A^{\mathcal{J}}$. Hence, it remains to show that there always exists such an η' for which additionally $(\delta, \eta') \in R^{\mathcal{J}}$. If $\eta = o^{\mathcal{I}}$ for some $o \in \mathsf{N}_I$, this is assured by (*). If R is non-simple, $\eta' := (\emptyset, \eta)$ has the desired property. If R is simple, letting $\eta' := (R, \eta)$ will satisfy the claim by (***). Thus, we have derived the validity of $A \sqsubseteq \exists R.B$ in KB.

Considering the axiom type $R_1 \sqsubseteq R_2$, assume $(\delta, \epsilon) \in R_1^{\mathcal{J}}$, whence we can conclude $(\mathsf{orig}(\delta), \mathsf{orig}(\epsilon)) \in R_1^{\mathcal{I}}$ which yields $(\mathsf{orig}(\delta), \mathsf{orig}(\epsilon)) \in R_2^{\mathcal{I}}$, as the considered axiom is contained in $\Theta^{\sqcap}(\mathrm{KB})$. In the case $\epsilon = \mathsf{orig}(\epsilon) = o^{\mathcal{I}}$ for some $o \in \mathsf{N}_I$, we find $(\delta, \epsilon) \in R_2^{\mathcal{J}}$ by (*). The same follows from (**) whenever R_2 is non-simple. It remains to consider the case that R_2 (and hence also R_1) is simple and $\epsilon = (x, \mathsf{orig}(\epsilon))$. The case $x = \emptyset$ can be excluded as then $(\delta, \epsilon) \notin R_1^{\mathcal{J}}$ by construction. Hence, $x = R$ for some simple role R. From $(\delta, \epsilon) \in R_1^{\mathcal{J}}$ and the construction of $\mathcal{J}$ we can conclude that $R_1 \in R^{\sqsubseteq}$. Via $R_1 \sqsubseteq R_2$ being in KB and the definition of $\cdot^{\sqsubseteq}$ follows $R_2 \in R^{\sqsubseteq}$, whereby the construction of $\mathcal{J}$ ensures $(\delta, \epsilon) \in R_2^{\mathcal{J}}$.

Considering the axiom type $R_1 \circ R_2 \sqsubseteq R_3$, let $(\delta, \epsilon) \in R_1^{\mathcal{J}}$ and $(\epsilon, \zeta) \in R_2^{\mathcal{J}}$, implying $(\mathsf{orig}(\delta), \mathsf{orig}(\epsilon)) \in R_1^{\mathcal{I}}$ as well as $(\mathsf{orig}(\epsilon), \mathsf{orig}(\zeta)) \in R_2^{\mathcal{I}}$. As the considered axiom is contained in $\Theta^{\sqcap}(\mathrm{KB})$, this entails $(\mathsf{orig}(\delta), \mathsf{orig}(\zeta)) \in R_3^{\mathcal{I}}$. By construction of $\mathcal{J}$ and due to the fact that R_3 is non-simple, for any δ' resp. ζ' originating from $\mathsf{orig}(\delta)$ resp. $\mathsf{orig}(\zeta)$ follows $(\delta', \zeta') \in R_3^{\mathcal{J}}$. Hence, in particular, this is the case for δ and ζ. Therefore, $R_1 \circ R_2 \sqsubseteq R_3$ is valid in $\mathcal{J}$.

Finally, we consider the axiom type $R_1 \sqcap R_2 \sqsubseteq R_3$. Hence, assume $(\delta, \epsilon) \in R_1^{\mathcal{J}}$ and $(\delta, \epsilon) \in R_2^{\mathcal{J}}$ which implies $(\mathsf{orig}(\delta), \mathsf{orig}(\epsilon)) \in R_1^{\mathcal{I}}$ as well as $(\mathsf{orig}(\delta), \mathsf{orig}(\epsilon)) \in R_2^{\mathcal{I}}$ (claim †).

In the case $\epsilon = \mathsf{orig}(\epsilon) = o^{\mathcal{I}}$ for some $o \in \mathsf{N}_I$, we find $(\mathsf{orig}(\delta), \mathsf{orig}(\epsilon)) \in R_3^{\mathcal{I}}$ as $\Theta^{\sqcap}(\mathrm{KB})$ contains the axiom $\exists R_1.\{o\} \sqcap \exists R_2.\{o\} \sqsubseteq \exists R_3.\{o\}$, thereby we obtain $(\delta, \epsilon) \in R_3^{\mathcal{J}}$. The same follows from (**) whenever R_3 is non-simple.

It remains to consider the case that R_3 is simple and $\epsilon = (x, \mathsf{orig}(\epsilon))$. Remember that R_1 and R_2 are required to be simple. Hence, the case $x = \emptyset$ can be excluded as then $(\delta, \epsilon) \notin R_1^{\mathcal{J}}$ (as well as $(\delta, \epsilon) \notin R_2^{\mathcal{J}}$) by construction. Hence, $x = R$ for some simple role R. From $(\delta, (R, \mathsf{orig}(\epsilon))) \in R_1^{\mathcal{J}}$ and $(\delta, (R, \mathsf{orig}(\epsilon))) \in R_2^{\mathcal{J}}$ as well as the construction of $\mathcal{J}$ we can conclude that $R_1, R_2 \in R^{\sqsubseteq}$. Yet then $\Theta^{\sqcap}(\mathrm{KB})$ also contains the axiom $R \sqsubseteq R_3$, whence (†) entails $(\mathsf{orig}(\delta), \mathsf{orig}(\epsilon)) \in R_3^{\mathcal{I}}$. Moreover, $R_3 \in R^{\sqsubseteq}$ guarantees $(\delta, \epsilon) \in R_3^{\mathcal{J}}$ via (***). □

The shown reduction – besides providing a way of using existing $\mathcal{EL}^{++}$ reasoning algorithms for reasoning in $\mathcal{EL}^{++}(\sqcap_s)$ – now gives rise to the complexity result for $\mathcal{EL}^{++}(\sqcap_s)$.

Theorem 3. *Satisfiability checking, instance retrieval, and computing class subsumptions for $\mathcal{EL}^{++}(\sqcap_s)$ knowledge bases is possible in polynomial time in the size of the knowledge base.*

Proof. Given an arbitrary $\mathcal{EL}^{++}(\sqcap_s)$ knowledge base KB, Proposition 2 ensures that it can be transformed in polynomial time into an equisatisfiable knowledge base KB′ in normal form. Again in polynomial time, we can compute the knowledge base $\Theta^{\sqcap}(\mathrm{KB}')$ that – by Proposition 3 – is equisatisfiable with KB′ (and hence also with KB). Finally, as $\Theta^{\sqcap}\mathrm{KB}'$ is an $\mathcal{EL}^{++}$ knowledge base, we can check satisfiability in polynomial time. □

We finish this section with some general remarks.

On the one hand, note that conjunction on roles enhances expressivity of $\mathcal{EL}^{++}$ significantly. For example, it allows for the following modelling features:

– Disjointness of two simple roles S, R. This feature, also provided by $\mathcal{SROIQ}$ as $\mathsf{Dis}(S, R)$, can be modelled in $\mathcal{EL}^{++}(\sqcap_s)$ by the axiom $\exists(S \sqcap R).\top \sqsubseteq \bot$.
– Atleast cardinality constraints on the right hand side of a GCI. The axiom $A \sqsubseteq {\geq}n\, R.B$ can be modelled by the axiom set $\{R_i \sqsubseteq R, A \sqsubseteq \exists R_i.B \mid 1 \leq i \leq n\} \cup \{\exists(R_i \sqcap R_j).\top \sqsubseteq \bot \mid 1 \leq i < j \leq n\}$ where $R_1, \ldots, R_n$ are new simple role names.

On the other hand, it is easy to see that incorporating more than just conjunction on simple roles into $\mathcal{EL}^{++}$ would render the respective fragment intractable at best:

Allowing conjunction on non-simple roles would even lead to undecidability as stated in Theorem 1 of [18].

Allowing disjunction or negation on simple roles would allow to model disjunction on concepts: for instance, the GCI $A \sqsubseteq B \sqcup C$ can be expressed by the axiom set $\{A \sqsubseteq \exists(R \sqcup S).\top, \exists R.\top \sqsubseteq B, \exists S.\top \sqsubseteq C\}$ or the axiom set $\{A \sqcap \exists R.\{o\} \sqsubseteq C, A \sqcap \exists\neg R.\{o\} \sqsubseteq B\}$ for new roles R, S and a new individual name o. Hence, any extension of $\mathcal{EL}^{++}$ into this direction would be ExpTime-hard [16].

7 DLP(⊓)

Description Logic Programs (DLP) constitutes a tractable knowledge representation formalism in the spirit of (Horn) logic programming [19]. Essentially, it consists of those $\mathcal{SHOIQ}$ axioms which can be naively translated into (non-disjunctive) Datalog, such that the original knowledge base and its translation are semantically equivalent. As such it represents the fragment of $\mathcal{SHOIQ}$ that can entail neither disjunctive information nor the existence of anonymous individuals as extensively studied in the context of Horn description logics [20]. Though rather complex syntactic definitions can be given to characterise all admissible axioms of such logics, we use a simpler definition comprising all essential expressive features of DLP without including all their syntactic varieties.

Definition 9. *Atomic roles of DLP are defined as in $\mathcal{SROIQ}$, including inverse roles. A* DLP body concept *is any $\mathcal{SROIQ}$ concept expression that includes only concept names, nominals, $\sqcap$, $\exists$, $\top$, and $\bot$. A* DLP head concept *is any $\mathcal{SROIQ}$ concept expression that includes only concept names, nominals, $\sqcap$, $\forall$, $\top$, $\bot$, and expressions of the form $\leq 1.C$ where C is a DLP body concept.*

A DLP knowledge base *is a set of Rbox axioms of the form $R \sqsubseteq S$ and $R \circ R \sqsubseteq R$, Tbox axioms of the form $C \sqsubseteq D$, and Abox axioms of the form $D(a)$ and $R(a, b)$, where $C \in \mathbf{C}$ is a body concept, $D \in \mathbf{C}$ is a head concept, and $a, b \in \mathsf{N}_I$ are individual names.*

DLP($\sqcap$) *knowledge bases are defined just as DLP knowledge bases, with the addition that conjunctions of roles may occur in DLP($\sqcap$) in all places where roles occur in DLP.*

Note that we do not have to distinguish between simple and non-simple roles for DLP.

In [20] it is shown that DLP is of polynomial worst-case complexity. This can be seen most easily by realising that DLP knowledge bases can be transformed in polynomial time (in the size of the knowledge base) into an equisatisfiable set of function-free first-order Horn rules (i.e. non-disjunctive Datalog rules) with at most three variables per formula. On the basis of this result, it is easy to show that DLP($\sqcap$) is also of polynomial complexity. We give a brief account of the argument.

Consider a DLP($\sqcap$) knowledge base K. We now perform the following transformation of K:

– For any role conjunction $R_1 \sqcap \cdots \sqcap R_n$ occuring in the knowledge base, replace the conjunction by a new role R, and add the axioms $R_1 \sqcap \cdots \sqcap R_n \sqsubseteq R$ and $R \sqsubseteq R_i$, for all $i = 1, \ldots, n$, to the knowledge base.

The resulting knowledge base is obviously equisatisfiable with K. It consists of two types of axioms: Axioms which are in DLP and axioms of the form $R_1 \sqcap \cdots \sqcap R_n \sqsubseteq R$. The latter axioms correspond to function-free Horn rules with only two variables. Hence, any DLP($\sqcap$) knowledge base can be transformed in polynomial time into an equisatisfiable set of function-free Horn rules.

Theorem 4. *Satisfiability checking, instance retrieval, and computing class subsumptions for DLP($\sqcap$) knowledge bases is possible in polynomial time in the size of the knowledge base.*

Proof. First note that instance retrieval and class subsumption can be reduced to satisfiability checking: Retrieval of instances for a class C is done by checking for all individuals a if they are in C – which in turn is reduced to satisfiability checking by adding the axioms $C \sqcap E \sqsubseteq \bot$ and $E(a)$ to the knowledge base, where E is a new atomic class name. Class subsumption $C \sqsubseteq D$ is reduced by adding the axioms $C(a)$, $E(a)$ and $D \sqcap E \sqsubseteq \bot$, for a new individual a and a new atomic class name E.

Now to check satisfiability of a DLP($\sqcap$) knowledge base, it is first transformed into an equisatisfiable set of function-free first-order Horn rules as mentioned above. The satisfiability of such a set of formulae can be checked in polynomial time, since any Horn logic program is semantically equivalent to its *grounding* (the set of all possible

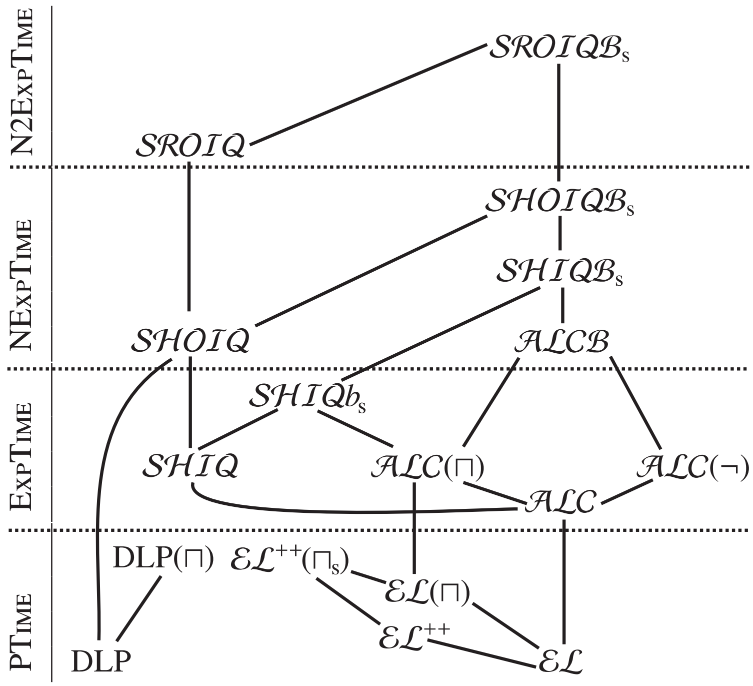

Fig. 2. Overview of complexities and expressivity relationships of DLs in the context of this paper.

ground instances of the given rules based on the occuring individual names). For a program with a bounded number n of variables per rule, this grounding is bounded by $r \times i^n$, where i is the number of individual names and r is the number of rules in the program. Finally, the evaluation of ground Horn logic programs is known to be P-complete. □

8 Conclusion

In our work, we have thoroughly investigated the reasoning complexities of DLs allowing for Boolean constructors on simple roles. We found that the expressive DLs $\mathcal{SROIQ}$ (being the basis of the forthcoming OWL 2 standard) and $\mathcal{SHOIQ}$ (the logical underpinning of OWL) can accommodate full Boolean role operators while keeping their reasoning complexities N2ExpTime and NExpTime, respectively. Likewise, the ExpTime-complete $\mathcal{SHIQ}$ can be safely extended by safe Boolean expressions. Finally, both the tractable fragments $\mathcal{EL}^{++}$ and DLP retain polynomial time reasoning complexity when adding just role conjunction, where in the case of DLP the role simplicity condition is not necessary. Figure 2 shows our findings integrated with other well-known complexity results relevant in this respect.

In particular we want to draw the reader's attention to the fact that – as opposed to hitherto proposed ways – the modelling of concept products and qualified role inclusions as presented in Section 1 does *not* automatically render the inferred roles non-simple. Moreover, due to the safety of the respective axiom, qualified role inclusions can even be modelled in $\mathcal{SHIQ}b_\mathrm{s}$.

Future work on that topic includes the further integration of the established results with our work on DL Rules [21], as well as the further investigation of the effects on complexity and decidability when allowing for Boolean constructors on non-simple roles.

Finally note that our results for $\mathcal{SHIQ}b_\mathrm{s}$, $\mathcal{EL}^{++}(\sqcap_\mathrm{s})$, and DLP($\sqcap$) provide direct ways for adapting existing reasoning algorithms for $\mathcal{SHIQ}$, $\mathcal{EL}^{++}$, and DLP, respectively. For $\mathcal{SROIQB}_\mathrm{s}$ and $\mathcal{SHOIQB}_\mathrm{s}$, however, setting up efficient algorithms seems less straightforward and represents another interesting direction of future research.

References

1. Rudolph, S., Krötzsch, M., Hitzler, P.: All elephants are bigger than all mice. In: Proc. 21st Int. Workshop on Description Logics (DL-08). (2008) To appear, extended online version available at `http://korrekt.org/page/Elephants`.
2. Horrocks, I., Kutz, O., Sattler, U.: The even more irresistible $\mathcal{SROIQ}$. In: Proc. of the 10th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR2006), AAAI Press (2006) 57–67
3. Baader, F., Calvanese, D., McGuinness, D., Nardi, D., Patel-Schneider, P., eds.: The Description Logic Handbook: Theory, Implementation and Applications. Cambridge University Press (2007)
4. Borgida, A.: On the relative expressiveness of description logics and predicate logics. Artif. Intell. **82** (1996) 353–367
5. Lutz, C., Sattler, U.: The complexity of reasoning with boolean modal logics. In Wolter, F., Wansing, H., de Rijke, M., Zakharyaschev, M., eds.: Advances in Modal Logics Volume 3, CSLI Publications, Stanford (2001)
6. Tobies, S.: Complexity Results and Practical Algorithms for Logics in Knowledge Representation. PhD thesis, RWTH Aachen, Germany (2001)
7. Pratt-Hartmann, I.: Complexity of the two-variable fragment with counting quantifiers. Journal of Logic, Language and Information **14** (2005) 369–395
8. Schmidt, R.A., Tishkovsky, D.: Using tableau to decide expressive description logics with role negation. In Aberer K. et al., ed.: ISWC/ASWC. Volume 4825 of LNCS., Springer (2007) 438–451
9. Lutz, C., Walther, D.: PDL with negation of atomic programs. Journal of Applied Non-Classical Logics **15** (2005) 189–213
10. Calvanese, D., Eiter, T., Ortiz, M.: Answering regular path queries in expressive description logics: An automata-theoretic approach. In: AAAI, AAAI Press (2007) 391–396
11. Glimm, B., Lutz, C., Horrocks, I., Sattler, U.: Answering conjunctive queries in the SHIQ description logic. Journal of Artificial Intelligence Research **31** (2008) 150–197
12. Kazakov, Y.: SRIQ and SROIQ are harder than SHOIQ. In: Proc. 21st Int. Workshop on Description Logics (DL-08), CEUR WS Proceedings (2008)
13. Demri, S., Nivelle, H.: Deciding regular grammar logics with converse through first-order logic. J. of Logic, Lang. and Inf. **14** (2005) 289–329

14. Rudolph, S., Krötzsch, M., Hitzler, P.: Terminological reasoning in SHIQ with ordered binary decision diagrams. In: Proceedings of the 23rd AAAI Conference on Artficial Intelligence (AAAI-08), AAAI Press (2008) Extended online version available at `http://www.aifb.uni-karlsruhe.de/WBS/sru/TR-rudolph-OBDD4SHIQ.pdf`.
15. Motik, B.: Reasoning in Description Logics using Resolution and Deductive Databases. PhD thesis, Universität Karlsruhe (TH), Germany (2006)
16. Baader, F., Brandt, S., Lutz, C.: Pushing the EL envelope. In: Proc. 19th Int. Joint Conf. on Artificial Intelligence (IJCAI 2005), Edinburgh, UK, Morgan-Kaufmann Publishers (2005)
17. Maier, D.: The Theory of Relational Databases. Computer Science Press (1983)
18. Krötzsch, M., Rudolph, S., Hitzler, P.: Conjunctive queries for a tractable fragment of OWL 1.1. In Aberer, K. et al., ed.: Proc. 6th Int. Semantic Web Conf. (ISWC 2007). Volume 4825 of LNCS., Springer (2007) 310–323
19. Grosof, B.N., Horrocks, I., Volz, R., Decker, S.: Description logic programs: combining logic programs with description logic. In: Proc. 12th Int. Conf. on World Wide Web (WWW 2003), ACM (2003) 48–57
20. Krötzsch, M., Rudolph, S., Hitzler, P.: Complexity boundaries for Horn description logics. In: Proc. 22nd AAAI Conf. (AAAI'07). (2007)
21. Krötzsch, M., Rudolph, S., Hitzler, P.: Expressive tractable description logics based on SROIQ rules. Technical report, Universität Karlsruhe (TH) (2008)